THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 19
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1 Legendre-Fenchel Conjugate

1. Given a convex set $A \subseteq \mathbb{R}^n$ and convex function $f : A \to \mathbb{R}$. We can extend

$$f(x) = \begin{cases} f(x) & \text{if } x \in A \\ \infty & \text{if } x \notin A \end{cases}$$

   and it is still convex. Therefore, we consider general function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$.

2. If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, then it is locally Lipschitiz on $\mathrm{ri}\ \mathrm{Dom}(f)$, and f is not constant on the boundary of $\mathrm{Dom}(f)$, where $\mathrm{Dom}(f) := \{x : f(x) \in \mathbb{R}\}$.

Definition 1. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

1. We define $\mathrm{epi}(f) := \{(x,t) : x \in \mathbb{R}^n,\ t \geq f(x)\} \subset \mathbb{R}^{n+1}$. Recall that f is convex if and only if $\mathrm{epi}(f)$ is convex.

2. f is lower semi-continuous (resp. upper semi-continuous) if

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}) \qquad (\text{resp.} \ \limsup_{x \to \bar{x}} f(x) \leq f(\bar{x}))$$

3. A convex function $f(x)$ is **proper** if $f(x) \in \mathbb{R} \cup \{+\infty\},\ \forall x \in \mathbb{R}^n$ and there exists $\hat{x} \in \mathbb{R}^n$ such that $f(x) \in \mathbb{R}$.

Example 1. When $n = 1$ and $A = [0,1]$.

- $f(x) = x\mathbb{1}_{x \in [0,1]} + \infty \mathbb{1}_{x \notin [0,1]}$ is lower semi-continuous.
- $f(x) = x\mathbb{1}_{x \in (0,1)} + \infty \mathbb{1}_{x \notin (0,1)}$ is **not** lower semi-continuous.

Proposition 1. *A function $f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is lower semicontinuous if and only if its epigraph* $\mathrm{epi}(f)$ *is closed (e.g. due to its emptiness).*

Proof. $(\Longrightarrow)$: Assume that f is lower semi-continuous and $(x_n, t_n)_{n \geq 1} \subset \mathrm{epi}(f)$ and $(x_n, t_n) \to (\bar{x}, \bar{t})$. We can deduce that $t_n \geq f(x_n)$, and thus the following relation holds:

$$\bar{t} = \lim_{n \to \infty} t_n \geq \liminf_{n \to \infty} f(x_n) \overset{\text{By the l.s.c}}{\geq} f(\bar{x})$$

It follows that $(\bar{x}, \bar{t}) \in \mathrm{epi}(f)$.
$(\Longleftarrow)$: On the other hand, let $\mathrm{epi}(f)$ be closed. If f is not lower semi-continuous, there exists sequence $(x_n)_{n \geq 1} \to \bar{x}$ such that $\lim_{n \to +\infty} f(x_n) < f(\bar{x})$. Define $\bar{t} := \lim_{n \to +\infty} f(x_n)$ and $t_n := f(x_n)$. Then, we have

$$(x_n, t_n) \in \mathrm{epi}(f) \implies (x_n, t_n) \to (\bar{x}, \bar{t})$$

but $\bar{t} < f(\bar{x})$. This means that $(\bar{x}, \bar{t}) \notin \mathrm{epi}(f)$. Contradiction to the closedness of $\mathrm{epi}(f)$. ☐

Immediately, we have the direct corollary followed by the consequences.

Corollary 2. *A function f is convex and lower semi-continuous if and only if its epigraph $\operatorname{epi}(f)$ is closed and convex.*

Lemma 3. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function so that $\operatorname{epi}(f)$ is convex. Then there exists a function $\operatorname{cl} f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ so that*

$$\operatorname{epi}(\operatorname{cl} f) = \overline{\operatorname{epi}(f)}$$

and $\operatorname{cl} f$ is lower semi-continous.

Proof. Let $g(x) := \lim_{r \searrow 0} \inf_{x:\|x'-x\| \leq r} f(x') \leq f(x)$ is lower semi-continuous. Then

1. $g(x) \leq f(x)$, for all $x \in \mathbb{R}^n$
2. $g(x) = f(x)$ for all $x \in \operatorname{ri} \operatorname{Dom}(f)$.

We can deduce that

$$\operatorname{Dom}(f) \subset \operatorname{Dom}(g) \subset \overline{\operatorname{Dom}(g)} \implies \operatorname{ri} \operatorname{Dom}(f) = \operatorname{ri} \operatorname{Dom}(g)$$

It follows that $\operatorname{Dom}(g) = \overline{\operatorname{Dom}(f)}$. □

— End of Lecture 19 —